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On the coexistence of laminar and turbulent flow in a narrow triangular duct

By P. C. BANDOPADHAYAY AND J. B. HINWOOD
Department of Mechanical Engineering, Monash University, Clayton, Victoria 3168, Australia

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1. Introduction

The possible coexistence of laminar and turbulent flow in a triangular duct had been a point of controversy among several authors. Since the nature of the flow–laminar or turbulent–has a major influence on the head loss and heat-transfer characteristics of the flow, resolution of this controversy is of practical as well as theoretical value.

From measurements made 24 and 35 hydraulic diameters from the inlet in 11·5° and 24·8° isosceles triangular ducts Eckert & Irvine (1956) concluded that a significant laminar zone existed in the apex of the duct while the remainder of the flow was turbulent. By flow visualization using smoke they showed that the size of this region decreased with increasing Reynolds number. At a number of cross-sections they also measured velocity profiles, which showed an abrupt change in the slope and curvature from $w \sim x^2$ to $w \sim x^{0\cdot 4}$, where w is the longitudinal velocity and x the transverse co-ordinate in the duct plane of symmetry. Flow visualization using dye confirmed the form of this relationship but gave a slightly earlier transition point.

Indirect support for this work comes from Kokorev *et al.* (1971), who applied Eckert & Irvine's data to predict heat transfer in triangular ducts with very good results.

On the other hand Hanks & Brooks (1970) severely criticized the dye technique and showed by using a birefringent fluid that the presence of a dye-injecting needle in the flow caused patterns of light and shade indicative of a disturbed flow. It is unfortunate that the technique of using birefringence has not been developed to the point where these patterns may be simply interpreted (Clayton & Massey 1967) although a rapidly changing irregular pattern undoubtedly shows the existence of a disturbed or turbulent flow. Hanks & Brooks interpreted the almost regular pattern of light and dark zones behind the probe as a generally turbulent flow rendered unstable by the probe and suggested that dye traces would show corresponding fluctuations. That their interpretations are not necessarily correct is shown by the photograph in figure 1 (plate 1), which shows a stable dye filament present together with an apparently disturbed birefringent pattern.

Cope & Hanks (1972) presented hot-wire anemometer traces intended to show that laminar and turbulent flow did not coexist and in fact that instability leading to turbulence first arose in Eckert & Irvine's laminar region, at a point predicted by using a stability parameter previously developed by Hanks. They showed that use of this stability parameter enabled them to predict the Reynolds number at which head loss in the duct ceased to obey the laminar law. Their paper presents much useful data and demonstrates the usefulness of Hanks' stability parameter, however, it leaves unresolved the question of the coexistence of laminar and turbulent flow as they gave no data from the apex region at high Reynolds numbers.

To resolve the questions raised by Hanks and his colleagues, to explain the nature of the laminar region and to predict its dimensions and structure the investigation reported here was conducted. A heat-transfer model similar to that of Kokorev *et al.* (1971) had been independently developed but the dimensions of the laminar region had to be known before it could be used in computation. This work arose out of the study of heat transfer in triangular ducts originated by Mr C. W. Ambrose, Senior Lecturer, Department of Mechanical Engineering, Monash University, and initially conducted under his supervision by the first author.

In the experiments to be described the velocities at points in a cross-section near the downstream end of a narrow isosceles triangular duct were measured using a hot-wire anemometer. These measurements confirm the results of Eckert & Irvine and are further supported by the analysis given in § 4, which predicts the point of transition from essentially laminar to turbulent flow.

2. Basic apparatus

The duct in which these studies were carried out is shown in figure 2. In section it is a 3·63° isosceles triangle of base 0·50 in., height 7·875 in. and length 15 ft 9 in., or 395 hydraulic diameters. The sides consist of steel plate of thickness $\frac{1}{8}$ in. held in the triangular cross-sectional shape by a heavy wooden frame with all the joints sealed with silicone sealant. Air is supplied by a centrifugal blower to a plenum chamber fitted with screens and thence via a rounded inlet to the duct. The duct is structurally isolated from the plenum chamber by a tightly fitting rubber sleeve just downstream from the inlet.

Pressure tappings are fitted at intervals along the base. The longitudinal pressure gradient was measured using a Betz differential micromanometer. The pressure gradients quoted were measured over the last 5 ft of the duct and were constant over at least this distance, giving indirect proof that the flow was fully established, although this is not a sensitive test of establishment.

The velocity measurements were all carried out approximately three hydraulic diameters upstream from the exit of the duct, i.e. 392 hydraulic diameters downstream from the inlet. The apparatus used was a specially made wire probe consisting of a platinum-coated tungsten wire 5 μm in diameter and 0·05 in. long welded onto a slender streamlined support. The hot wire was connected to a Disa 55D05 constant-temperature anemometer set and thence to an oscilloscope. The calibration of the wire was $\sqrt{u} = 5{\cdot}2(E/E_0)^2 - 4{\cdot}85$ for $u > 2$ ft/s, where E is the voltage at a velocity of u ft/s, and is E_0 at $u = 0$.

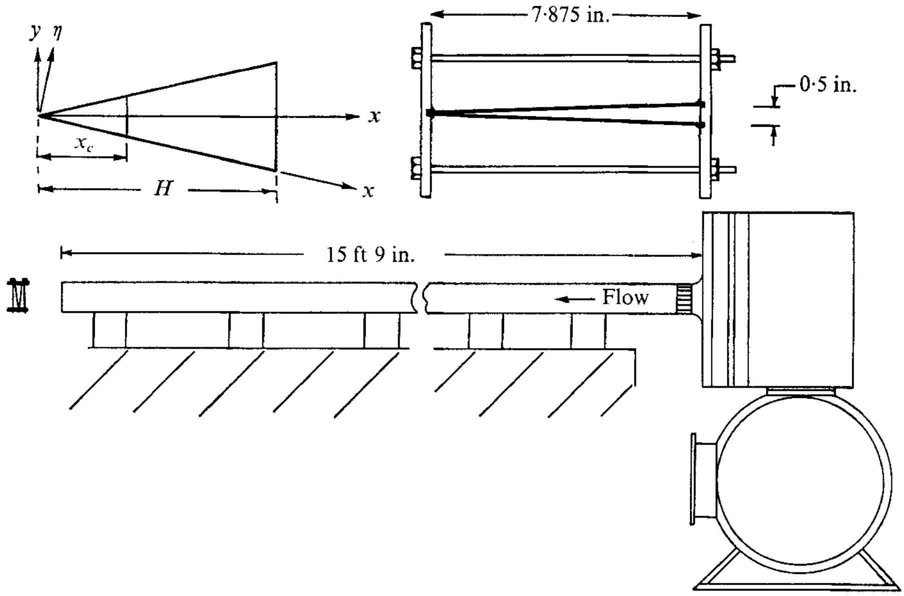

FIGURE 2. Line sketch of the duct and definition of co-ordinate system.

3. Experimental results

3.1. *The coexistence of laminar and turbulent flow*

Typical hot-wire anemometer traces are shown in the composite photograph in figure 3 (plate 2). The ordinate of this figure is the longitudinal pressure gradient dp/dz. Since this is a monotonic although not a linear function of the mean velocity and Reynolds number, both of these quantities increase towards the top of the figure. The abscissa of figure 3 is the distance x along the plane of symmetry from the apex.

It may be seen from these traces that for small values of dp/dz and x near the origin the flow is steady, and may be termed 'laminar', and for large enough values of dp/dz or x the flow is unsteady at all times, i.e. 'turbulent'. Since weak fluctuations are still observed in the 'laminar' region a more apt title might be 'viscous layer', but 'laminar' will be used for convenience.

Consider the traces shown on the horizontal line $dp/dz = 0{\cdot}257\,\text{lb/ft}^3$. For values of x less than 2·5 the flow is laminar, and for values of x greater than 3·0 the flow is turbulent. Similar results are found for $dp/dz = 0{\cdot}159\,\text{lb/ft}^3$. The conclusion is that over some range of pressure gradients laminar and turbulent flows coexist in the triangular duct.

3.2. *The 'laminar' region*

Between the essentially laminar and turbulent regions in the duct there was found to be a region in which the flow showed intermittently one sort of behaviour or the other, suggesting analogy with the edge of a turbulent submerged jet. Since an objective measure of the boundary between laminar and turbulent regions is required some quantitative measure of intermittency must be made.

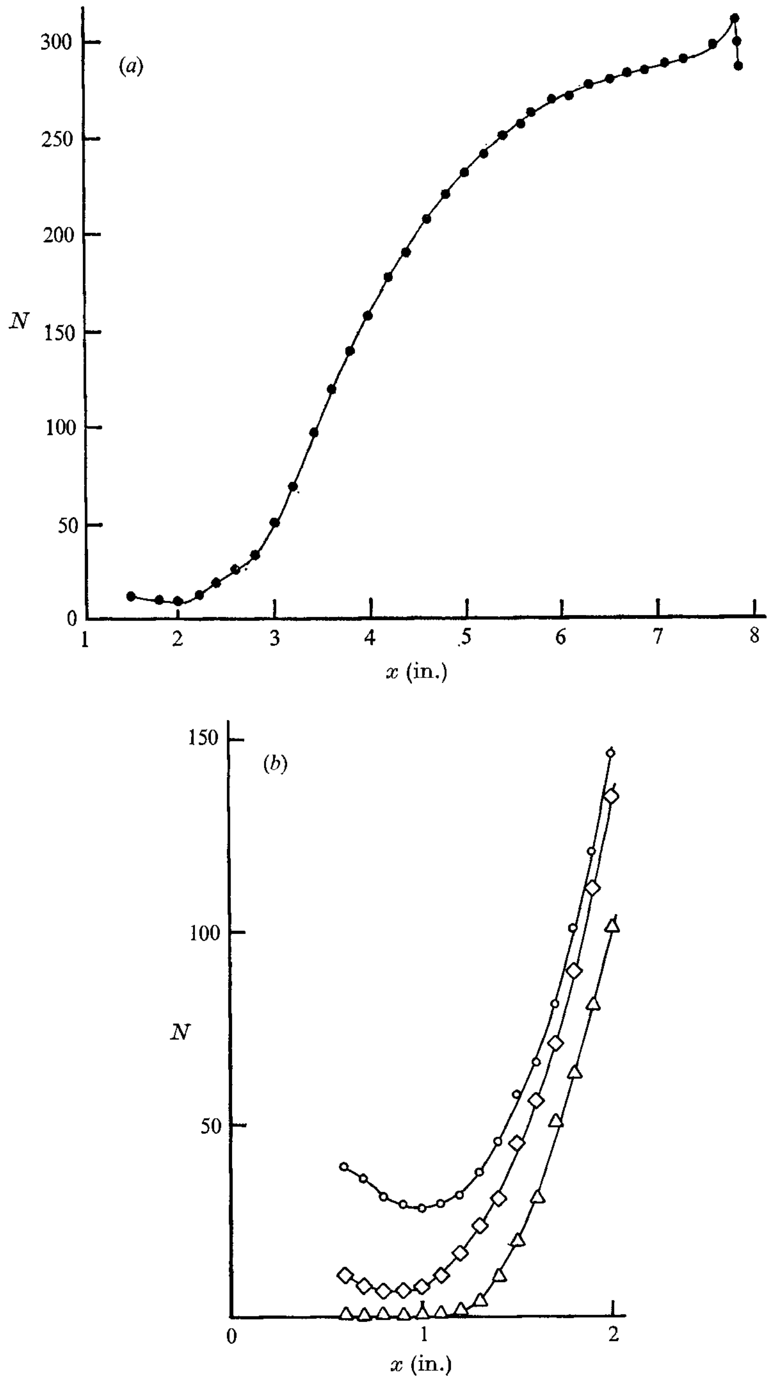

FIGURE 4. Average number of times N per second that signal voltage exceeded the local time-averaged voltage, as a function of position x on the plane of symmetry in a cross-section, (a) $dp/dz = 0{\cdot}257\ \mathrm{lb/ft^3}$; base of duct at $x = 7{\cdot}875$ in. (b) $dp/dz = 0{\cdot}265\ \mathrm{lb/ft^3}$; ○, gate at 4·4 mV; ◇, 6·0 mV; △, 10·0 mV.

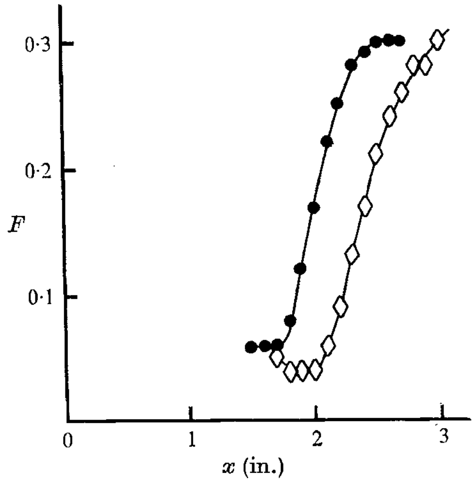

FIGURE 5. Fraction of time F that signal voltage exceeded gate level; gate at 4·65 mV. $\diamond$, dp/dz = 0·253 lb/ft^3; $\bullet$, dp/dz = 0·392 lb/ft^3. Intermittency $\approx 2F$.

A zero-crossing meter was used to study the hot-wire anemometer signals: the number of times the signal voltage crossed the mean voltage was counted over a period of 10 s and the average number of crossings per second was found. This figure, N, was plotted against x for a given pressure gradient in figure 4 (a). It may be seen that, at a certain value of x, N attained a minimum value and just past this point, at $x = x_c$, N began to increase rapidly. It is not known why N rises as x is reduced below x_c, but the value of N remains very small. The experiments were repeated over a 14 fold range of pressure gradients and the x_c values obtained are given in figure 6, which shows that over the range of pressure gradients employed

$$x_c \sim (-dp/dz)^{-\frac{1}{3}}.$$

The anemometer signals were also passed through a gate circuit, the level of which could be set at a predetermined voltage with respect to the local time-averaged voltage. The average number of times per second that the magnitude of the signal voltage exceeded the gate voltage was determined. This value of N was also plotted against x, as shown in figure 4 (b). Each point on the figure is the mean of four samples each of 10 s duration. For a fixed pressure gradient the value of N at a given point in the duct decreased with gate level, as is to be expected, but the value of x_c was found to be independent of gate level and equal to the value obtained from figure 4 (a) for a gate level equal to the mean voltage.

The ratio of N at a point to the value obtained in the fully turbulent regions gives a very approximate measure of the intermittency. Since the intermittency is more readily comparable from one experimental situation to another than zero crossings, and is more directly related to the observed nature of the anemometer traces in figure 4, a simple intermittency meter was constructed and was used to measure the values of intermittency given in figure 5. The construction and performance characteristics of the meter are discussed in a paper in preparation. The division between laminar and turbulent flow was again chosen to be the point at which the curve started to rise rapidly. The values of x_c obtained from plots such as figure 5 were the same as those already obtained, within experimental error.

3.3. *Accuracy of the measurement of x_c*

By inspection of figure 4(*a*), the determination of x_c from the graph is subject to a maximum error of $\pm 0{\cdot}1$ in. and since this far exceeds the errors in positioning the hot-wire probe in the vertical (x) direction this will be taken as the error in x_c.

Errors in positioning the hot-wire probe in the transverse (y) direction will also contribute to the error in determining x_c. For several values of x, the probe was displaced and repositioned but the spread of these readings did not differ from those obtained by repeated readings with the probe fixed in one position This finding was supported by taking some measurements off the plane of symmetry (i.e. $y = 0$), which showed that the nature of the flow varied very little with y near $y = 0$. Hence the error in x_c due to this cause may be neglected in comparison with the $\pm 0{\cdot}1$ in.

The static pressure gradient was determined with a Betz differential micromanometer and is subject to errors of $\pm 0{\cdot}01$ mm of water, with a negligible error in the distance over which the pressure gradient was determined. Hence the maximum probable error in the pressure gradients is $\pm 4 \times 10^{-4}\,\mathrm{lb/ft^3}$.

4. Thickness of the 'laminar' layer

The thickness of the laminar region in the apex of the duct will now be determined subject to the assumptions below. The co-ordinates x, y and z are as before with z parallel to the axis of the duct, and the co-ordinate χ is measured from the apex along one side of the duct and η is perpendicular to χ. Transition occurs at $\chi = \chi_c$. The following assumptions are made.

(*a*) The flow is fully developed, i.e. $dp/dz =$ constant.

(*b*) There is no secondary flow, so that the velocity vector is always in the $+z$ direction.

(*c*) From the results presented in §3, the flow in the duct consists of two regions, one 'laminar', the other 'turbulent'. This assumption means that the transition region, which shows intermittent turbulence, is divided into 'laminar' and 'turbulent' parts, a simplification which, although it has often been used in fluid dynamics and heat-transfer studies, has no fundamental basis.

(*d*) Transition from laminar to turbulent flow occurs when, at some point on a line perpendicular to the wall, the critical value of a stability parameter is exceeded.

Following the usual approach applied to the study of the turbulent boundary layer we assume that instability of the laminar region, leading to turbulence, will arise when a critical value of $k = y_{\max} u_* / \nu$ is exceeded. Here u_* is the wall shear velocity and ν the kinematic viscosity.

To evaluate k the wall shear stress is found by using the laminar velocity profile of Eckert & Irvine (1956):

$$w = -\frac{1}{2\mu}\frac{dp}{dz}(x^2 \tan^2\alpha - y^2)/(1-\tan^2\alpha), \tag{1}$$

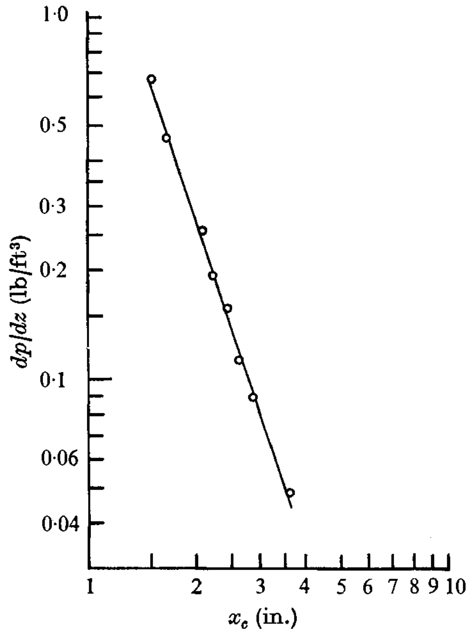

FIGURE 6. Location of transition point x_c as a function of pressure gradient dp/dz along the duct; log scales.

$$\tau_0 = \mu\left(\frac{\partial w}{\partial \eta}\right)_{\eta=0}$$

$$= -x\frac{dp}{dz}(\tan^2\alpha + 1)\sin\alpha/(1-\tan^2\alpha), \tag{2}$$

hence
$$x_c = \left(\frac{\mu^2}{\rho}\right)^{\frac{1}{3}}\left(-\frac{dp}{dz}\right)^{-\frac{1}{3}}\frac{[k^2(1-\tan^2\alpha)/\cos\alpha(1+\tan^2\alpha)]^{\frac{1}{3}}}{\tan\alpha}. \tag{3}$$

From the experimental results presented in § 3, values of dp/dz and x_c for a single value of α and μ^2/ρ were obtained and are plotted in figure 6. Values of k computed from (3) produced small but random scatter with the mean value of 26·1, which compares well with the range of 26–30 given in the literature.†

To facilitate comparison with other studies, (3) may be written in terms of the Reynolds number Re, where $Re = dV/\nu$ and d is the hydraulic diameter and V the average velocity in the duct.

From measurements of discharge and pressure gradient the following relationship between the resistance coefficient f and the Reynolds number was obtained and is plotted in figure 7:

$$f = 0{\cdot}25/Re^{\frac{1}{4}}.$$

† This result may be extended to the use of other stability criteria since it can be shown by dimensional reasoning that if $x_c = \phi_1(\mu, \rho. dp/dz, \tau, \Phi(w), \alpha)$, where ϕ denotes some function and $\Phi(w)$ is the maximum value, for a given value of x, of the velocity or a velocity gradient or a product of velocity or velocity gradients, then

$$x_c = (\mu^2/\rho)^{\frac{1}{3}}(-dp/dz)^{-\frac{1}{3}}\phi_3(\alpha).$$

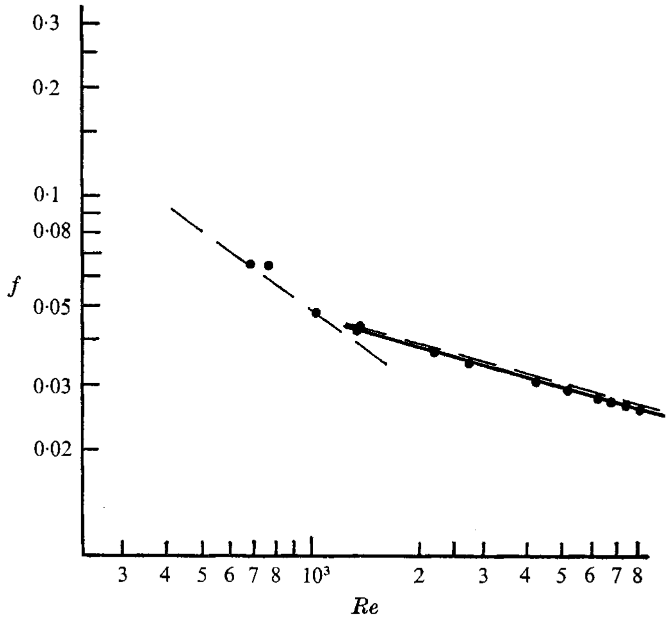

FIGURE 7. Measured friction factor *vs.* Reynolds number. – – –, α = 2°, Carlson & Irvine (1961); ●, α = 1·93° (duct of figure 2).

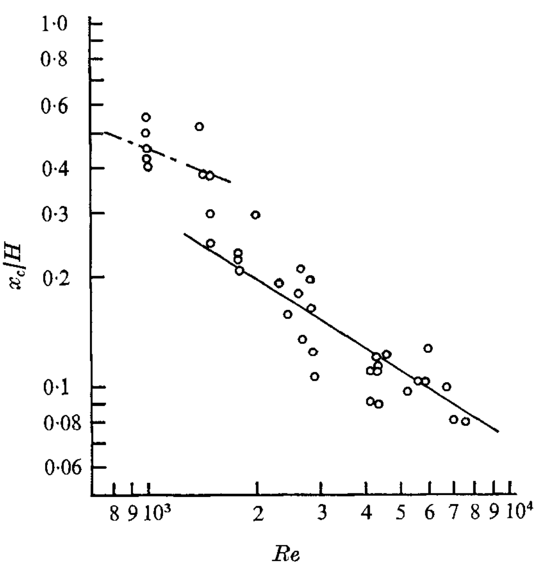

FIGURE 8. Location of transition point as a function of Reynolds number, data of Eckert & Irvine (1956) with a line of slope −0·58.

This is a particular case of $f = C_1/Re^{\frac{1}{4}}$ found by Blasius for flow through circular tubes, Leutheusser (1963) for flow in square ducts and Eckert & Irvine (1956) for triangular ducts. The value of C_1 depends on the duct shape, and in the case of triangular ducts this means that C_1 is a function of α. From the definition of f and the above expression

$$-dp/dz = C_1(\alpha)\,\rho V^2/2d\,Re^{\frac{1}{4}}.$$

Hence $$x_c = d\,Re^{-\frac{7}{12}}[2k^2(1-\tan^2\alpha)/C_1(\alpha)\cos\alpha(1+\tan^2\alpha)]^{\frac{1}{3}}/\tan\alpha.$$

Thus x_c is proportional to $Re^{-\frac{7}{12}}$. Although the original data of Eckert & Irvine (1956) have large scatter, this trend is quite evident from their figure, which is replotted here as figure 8.

The derivation above has correctly predicted the thickness of the laminar zone by regarding it as a viscous sublayer. The presence of the corner leads to interaction of the boundary layers on each of the sides of the duct forming a significant laminar region in the corner while the remainder of the flow is turbulent with thin viscous sublayers against the walls.

Support for this explanation may be drawn from analyses and measurements of turbulent flows in ducts with a corner, although not of triangular form. For example, the flow in the open-sided right-angled triangular duct formed by two infinite planes has been analysed by Gersten (1959), who showed that there was significant thickening of the boundary layer in the corner, leading to an appreciable viscous sublayer. Leutheusser's experimental velocity profiles in a rectangular duct show a similar effect.

5. Conclusions

From the above observations it may be seen that the two-regime flow described by Eckert & Irvine occurs over a range of Reynolds numbers. The 'laminar' region is nothing but a greatly thickened viscous sublayer formed by interaction of the boundary layers on the side walls of the duct. The use of boundary-layer or other stability criteria have enabled the thickness of the 'laminar' region to be predicted.

REFERENCES

Carlson, L. W. & Irvine, T. F. 1961 *Trans. A.S.M.E., J. Heat Transfer*, **83**, 441–444.

Clayton, B. R. & Massey, B. S. 1967 *J. Sci. Instrum.* **44**, 2–11.

Cope, R. C. & Hanks, R. W. 1972 *Ind. Engng Chem. Fund.* **11**, 106–117.

Eckert, E. R. G. & Irvine, T. F. 1956 *Trans. A.S.M.E.* **78**, 709–718.

Gersten, F. 1959 Corner interference effects. *Inst. für Aerodynamics Braunschweigh. Publ.* DFL-Bericht Nr-108.

Hanks, R. W. & Brooks, J. C. 1970 *A.I.Ch.E. J.* **16**, 483–489.

Kokorev, L. S., Korsun, A. S., Kostyunin, B. N. & Petrovichev, V. I. 1971 *Heat Transfer, Sov. Res.* **3**, 56–65.

Leutheusser, H. J. 1963 *J. Hyd. Div. Proc. A.S.C.E.* **89**, 1–19.

FIGURE 1. Wake of a 0·048 in. diameter probe in the laminar flow of a birefringent liquid, showing a stable dye trace and a disturbed birefringent pattern.

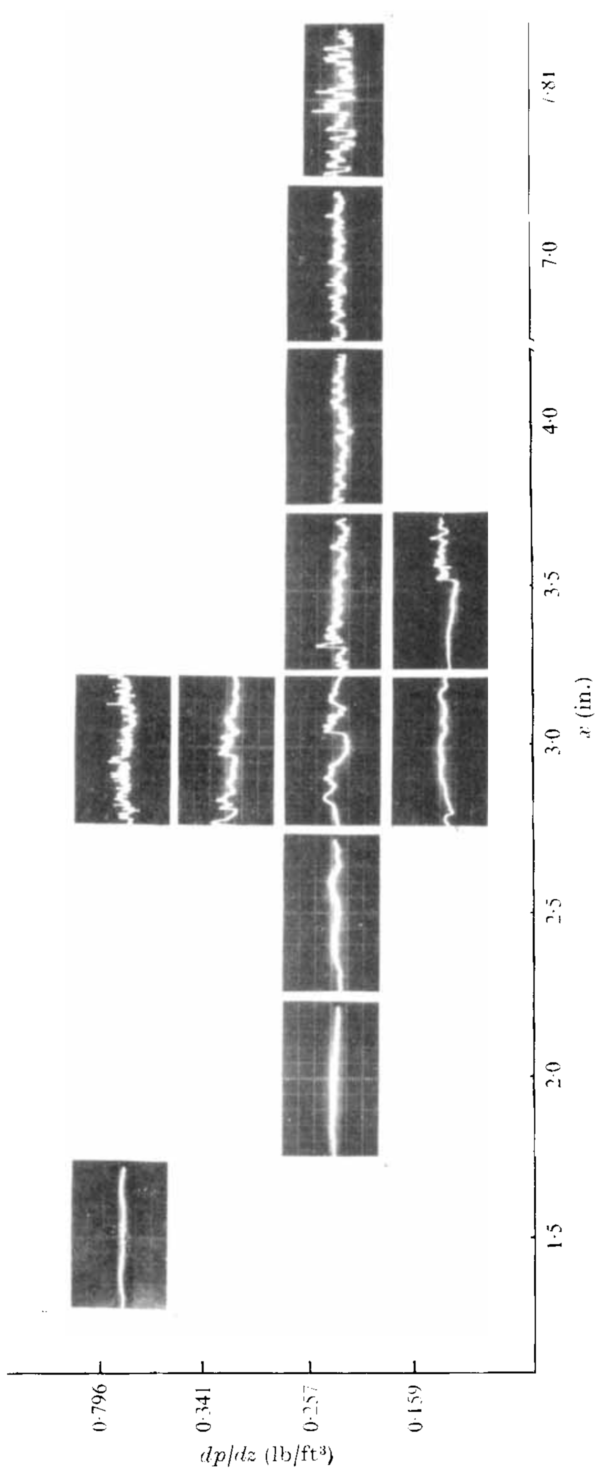

Figure 3. Composite photograph of hot-wire traces at various positions on the centre-line of the duct cross-section for different pressure gradients.

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